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# STATIC CYLINDRICALLY SYMMETRIC SOLUTIONS OF EINSTEIN'S EQUATIONS

***M. A. Kauser[1] & Q. Islam[2]***

*[1]Research Scholar, Department of Mathematics, Chittagong University of Engineering and Technology, Bangladesh*

*[2]Research Scholar, Jamal Nazrul Islam Research Center for Mathematical and Physical Sciences, University of Chittagong, Bangladesh*

## *ABSTRACT*

*In recent years, a number of solution generating techniques for spherically symmetric perfect fluid solutions of Einstein's equations have been invented. Besides, solutions with cylinderical symmetry are much less studied because of the complexity of calculations involving the equations for cylinderically spacetime. For our kith interest in the cylindrically symmetric static perfect fluid solutions of Einstein's equations, we have provided an algorithm and find a new realistic solution.*

***KEYWORDS:*** *Spherical Symmetry, Cylinderical Symmetry, Tangential Gauge, Arc-length Gauge, Metric Function, Axis of Symmetry, Minkowski Space*

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## 1. INTRODUCTION

Spherically symmetric solutions of Einstein's equations have attracted the interests of researchers working in this field. This is because phenomena such as black-holes, neutron stars etc. have been found in the class of solutions [1-8] with spherical symmetry. As a result many such solutions have been discovered. In contrast, solutions with cylindrical symmetry are much less studied. Whereas in the case of spherical symmetry, one is to solve for two unknown metric functions in the case of cylindrical symmetry, other is to solve for three unknown metric functions. According to Birkhoff's theorem, exterior solution of any spherically symmetric source is uniquely determined by the Schwarzschild solution. On the other hand, there is no Birkoff's theorem for space-times with cylindrical symmetry. There exist infinitely many exterior solutions for space-times with cylindrical symmetry. Any cylindrically symmetric interior solution must be joined with any one of those exterior solutions at the boundary of the cylindrically symmetric source. The general form of the spherically symmetric space-time metric in spherical coordinates is unique. In contrast, general form of the cylindrically symmetric space-time metric is not unique so that the same solution may look different for different definitions of the radial coordinate in a cylindrical coordinate system.

The rest of this paper is organized in the following way. In Section-2, we write down the general form of the static cylindrically symmetric space-time metric in the tangential gauge and derive all vacuum solutions. In Section-3, we write down the form of the metric in the arc-length gauge and derive some perfect fluid solutions for specific choices of one of the three metric functions. This is mainly a review. In Section-4, we have derived the field equations in the tangential gauge. Main result of this chapter is provided in Section-5. Here, we have shown that the field equations can be reduced to a pair of simultaneous Riccati type differential equations whose general solution depends on the specification of a particular solution f(r). It is also shown that, a class of physically acceptable solutions can be generated if f(r) satisfies the condition f (0) = 0. An algorithm is provided for generating all such solutions. In Section-6, the algorithm is illustrated by generating a new realistic solution. Finally in Section-7, some concluding remarks are given.

## 2. STATIC CYLINDRICALLY SYMMETRIC VACUUM SOLUTIONS

The general static cylindrically symmetric space-time metric can be written as

$$ds^2 = -e^{2\varphi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + e^{2\Omega(r)}d\theta^2 + e^{2\psi(r)}dz^2 \tag{1}$$

Here z denotes the coordinate along the central axis of symmetry, r denotes the radial coordinate which has value zero on the axis of symmetry and which increases as one moves away from the axis of symmetry, $\theta$ denotes the coordinate which measures an angle around the axis of symmetry.

Let us choose the gauge by defining the new radial coordinate $r'$ such that $(r')^2 = e^{2\Omega(r)}$. Then the metric (1) reduces to

$$ds^2 = -e^{2\varphi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2 + e^{2\psi(r)}dz^2 \tag{2}$$

Where the prime on r has been omitted. The form in which metric (2) is written is called tangential gauge. There are many different conventions for defining the radial coordinate. For example, the radial coordinate can be defined in such a way that metric (1) takes the form

$$ds^2 = -e^{2\varphi(r)}dt^2 + dr^2 + e^{2\Omega(r)}d\theta^2 + e^{2\psi(r)}dz^2 \tag{3}$$

The form in which metric (3) is written is called arc-length gauge.

We are interested to find vacuum solutions. In this case, Einstein's equations reduce to $R_{\alpha\beta} = 0$. We find it convenient to work in the tangential gauge in which the metric has the form (2). Nonzero components of $R_{\alpha\beta}$ for the metric (2) are given by [9],

$$R_{00} = e^{2(\varphi-\Lambda)}\left\{\varphi'' + (\varphi')^2 - \varphi'\Lambda' + \frac{\varphi'}{r} + \Psi'\varphi'\right\} \tag{4}$$

$$R_{rr} = -\varphi'' - (\varphi')^2 + \varphi'\Lambda' + \frac{\Lambda'}{r} - \Psi'' - (\Psi')^2 + \Lambda'\Psi' \tag{5}$$

$$R_{\theta\theta} = e^{-2\Lambda} r\left(\Lambda' - \varphi' - \Psi'\right) \tag{6}$$

$$R_{zz} = -e^{2(\Psi-\Lambda)}\left\{\Psi'' + (\Psi')^2 - \Psi'\Lambda' + \Psi'\varphi' + \frac{\Psi'}{r}\right\} \tag{7}$$

Where primes denote differentiations with respect to r. Einstein's vacuum field equations $R_{\alpha\beta} = 0$ then provide the following system of equations,

$$\Lambda' = \varphi' + \Psi' \tag{8}$$

$$\varphi'' + \frac{\varphi'}{r} = 0 \tag{9}$$

$$\Psi'' + \frac{\Psi'}{r} = 0 \tag{10}$$

$$\varphi'\Psi' + \frac{\varphi'}{r} + \frac{\Psi'}{r} = 0 \tag{11}$$

Thus we have four equations for three unknown functions. The system of equations (8) - (10) has the solution

$$\varphi = \log(c_1 r^a)\,,\ \Psi = \log(c_2 r^b)\,,\ \Lambda = \log(c_3 r^{a+b}) \qquad \text{(12a, b, c)}$$

Inserting (12a, b, c) into (11) we obtain the constraint

ab + a +b = 0 (13)

Therefore all vacuum static cylindrically symmetric solutions of Einstein's equations are given by

$$ds^2 = -c_1^{\ 2} r^{2a} dt^2 + c_3^{\ 2} r^{2(a+b)} dr^2 + r^2 d\theta^2 + c_2^{\ 2} r^{2b} dz^2 \tag{14}$$

Subject to the constraint (13). The constants $c_1^{\ 2}$ and $c_2^{\ 2}$ can be absorbed by rescaling t and z. The constant $c_3^{\ 2}$ can be absorbed by rescaling r, which affects the $d\theta^2$ term by bringing out another constant $k^2$ in its coefficient. The constant $k^2$ can be absorbed by rescaling $\theta$ which redefines its range from 0 to some angle $\alpha^*$. Therefore, all vacuum static cylindrically symmetric solutions of Einstein's equations in the tangential gauge are given by

$$ds^2 = -r^{2a} dt^2 + r^{2(a+b)} dr^2 + r^2 d\theta^2 + r^{2b} dz^2 \tag{15}$$

Where a and b are restricted by the constraint (13) and where $0 \le \theta \le \alpha^*$, where $\alpha^*$ may or may not be equal $2\pi$. If a = b = 0, constraint (13) is satisfied. Then (15) reduces to

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2 \tag{16}$$

Metric (16) represents the ordinary Minkowski space if it happens that $\alpha^* = 2\pi$. Otherwise it represents a cone solution.

Any interior cylindrically symmetric solution must be joined smoothly with any one of the 1-parameter family of solutions (15) at the boundary of the cylindrically symmetric source.

## 3. STATIC CYLINDRICALLY PERFECT FLUID SOLUTIONS

Previously, perfect fluid solutions with cylindrical symmetry have been discussed, among others, by Evan [10], Bronnikov [11] and Sharif [12]. Evan and Bronnikov used various equations of state which can be written as $\rho = \gamma p$ for specific values of $\gamma$ as well as energy conservation equation. Sharif found some solutions by specifying one of the three metric functions. They used the form of the metric in the arc-length gauge. In Section-4.4, we will show that, using the form of the metric in the tangential gauge, it is possible to find all static cylindrically symmetric perfect fluid solutions in closed form. Before that, let us see how solutions are derived in [12] using the form of the metric in the arc-length gauge, which we rewrite for convenience in the following way,

$$ds^2 = -e^{\gamma(r)}dt^2 + dr^2 + e^{\lambda(r)}d\theta^2 + e^{\mu(r)}dz^2 \tag{17}$$

If the matter content is perfect, fluid Einstein's equations provide the system of equations [11],

$$8\pi\rho = -e^{-\frac{\lambda}{2}}\left(e^{\frac{\lambda}{2}}\right)'' - e^{-\frac{\mu}{2}}\left(e^{\frac{\mu}{2}}\right)'' - \frac{\lambda'\mu'}{4} \tag{18}$$

$$8\pi p = e^{-\frac{\mu}{2}}\left(e^{\frac{\mu}{2}}\right)'' + e^{-\frac{\gamma}{2}}\left(e^{\frac{\gamma}{2}}\right)'' + \frac{\mu'\gamma'}{4} \tag{19}$$

$$8\pi p = e^{-\frac{\gamma}{2}}\left(e^{\frac{\gamma}{2}}\right)'' + e^{-\frac{\lambda}{2}}\left(e^{\frac{\lambda}{2}}\right)'' + \frac{\lambda'\gamma'}{4} \tag{20}$$

$$8\pi p = \frac{1}{4}\left(\lambda'\mu' + \mu'\gamma' + \lambda'\gamma'\right) \tag{21}$$

To solve the above system of equations the cases (A) $\gamma = 0$, (B) $\lambda = 0$ and (C) $\mu = 0$ are considered.

**Case A:** In this case equations (19) – (21) can be expressed as

$$\frac{y''}{y} = \frac{z''}{z}, \quad \frac{y''}{y'} = \frac{z'}{z} \quad \text{and} \quad \frac{z''}{z'} = \frac{y'}{y} \tag{22a, b, c}$$

Where $y = e^{\frac{\mu}{2}}$, $z = e^{\frac{\lambda}{2}}$ and primes denote derivatives with respect to r. From (22b) and (22c) we get

$$y' = k_1 z, \quad z' = k_2 y \tag{23a, b}$$

Equations (23a) and (23b) imply

$$\frac{y''}{y} = \frac{z''}{z} = k_1 k_2 \tag{24}$$

Which imposes a constraint. Using (23b) in (23a) we get

$$y'' - (k_1 k_2) y = 0 \tag{25}$$

Equation (25) can be solved for y. Knowing y(r), z(r) can be found from (23b). Four cases arise, (i) $k_1 k_2 < 0$, (ii) $k_1 k_2 > 0$, (iii) $k_1 \neq 0$, $k_2 = 0$ and (iv) $k_1 = 0$, $k_2 \neq 0$.

(i) In this case we get the solution

$$e^{\frac{\mu}{2}} = k_3 \cos(\sqrt{-k_1 k_2}\, r + k_4)$$

$$e^{\frac{\lambda}{2}} = k_3 \sqrt{-\frac{k_2}{k_1}} \sin(\sqrt{-k_1 k_2}\, r + k_4)$$

(ii) In this case we obtain

$$e^{\frac{\mu}{2}} = k_4 \cosh(\sqrt{k_1 k_2}\, r)$$

$$e^{\frac{\lambda}{2}} = k_4 \sqrt{\frac{k_2}{k_1}} \sinh(\sqrt{k_1 k_2}\, r + k_5)$$

(iii) $e^{\frac{\mu}{2}} = c_1 r + c_2$

$$e^{\frac{\lambda}{2}} = c_3$$

(iv) $e^{\frac{\mu}{2}} = c_3$

$$e^{\frac{\lambda}{2}} = c_1 r + c_2$$

For each of these solutions we get

$$\rho = -\frac{3}{8\pi} \mathrm{k_1 k_2}, \quad p = \frac{1}{8\pi} \mathrm{k_1 k_2}, \quad \rho + 3p = 0$$

We see that solution with $\mathrm{k_1 k_2} < 0$ has a positive energy density while with $\mathrm{k_1 k_2} > 0$ has a negative energy density. If $\mathrm{k_1 k_2} = 0$, the solution is trivial.

**Case B:** In this case equations (19) – (21) reduce to

$$\left(e^{\frac{\gamma}{2}}\left(e^{\frac{\mu}{2}}\right)'\right)' = 0 \tag{26}$$

$$e^{-\frac{\mu}{2}}\left(e^{\frac{\mu}{2}}\right)'' + e^{-\frac{\gamma}{2}}\left(e^{\frac{\gamma}{2}}\right)'' = 0 \tag{27}$$

$$\left(e^{-\frac{\mu}{2}}\left(e^{\frac{\gamma}{2}}\right)'\right)' = 0 \tag{28}$$

In this case $\rho$ and $p$ are given by

$$\rho = \frac{1}{8\pi}\mathrm{k}_1\mathrm{k}_2 e^{-\gamma} = p$$

We find that both $\rho$ and $p$ depend on $\gamma(r)$.

**Case C:** Case C is trivial and can be solved by replacing $\mu$ in Case B by $\lambda$. This yields the Evan's solution [10].

## 4. FIELD EQUATIONS IN THE TANGENTIAL GAUGE

We are interested in finding static cylindrically symmetric internal solutions when the matter content is perfect fluid. For this, we are to solve the equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \tag{29}$$

We find it convenient to use the form of the metric in the tangential gauge which we rewrite for convenience,

$$ds^2 = -e^{2\varphi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2 + e^{2\psi(r)}dz^2 \tag{2}$$

Nonzero components of the Einstein's tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ for the metric (2) are given by [9]

$$G_{tt} = e^{2(\varphi-\Lambda)}\left(-\Psi'' - \Psi'^2 + \psi'\Lambda' + \frac{\Lambda'}{r} - \frac{\Psi'}{r}\right)$$

$$G_{rr} = \Psi'\varphi' + \frac{\varphi'}{r} + \frac{\Psi'}{r}$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda}\left(\varphi'' + \varphi'^2 - \varphi'\Lambda' + \Psi'' + \Psi'^2 - \Psi'\Lambda' + \Psi'\varphi'\right)$$

$$G_{zz} = e^{2(\Psi-\Lambda)}\left(\varphi'' + \varphi'^2 - \varphi'\Lambda' - \frac{\Lambda'}{r} + \frac{\varphi'}{r}\right)$$

Since the matter content is perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}$$

Nonzero components of $T_{\mu\nu}$ are found to be

$T_{tt} = \rho e^{2\varphi}$, $T_{rr} = pe^{2\Lambda}$, $T_{\varphi\varphi} = pr^2$, $T_{zz} = pe^{2\Psi}$

Using these in equation (29) we obtain the equations

$$8\pi\rho = e^{-2\Lambda}\left(-\Psi'' - \Psi'^2 + \psi'\Lambda' + \frac{\Lambda'}{r} - \frac{\Psi'}{r}\right) \tag{30}$$

$$8\pi p = e^{-2\Lambda}\left(\Psi'\varphi' + \frac{\varphi'}{r} + \frac{\Psi'}{r}\right) \tag{31}$$

$$8\pi p = e^{-2\Lambda}\left(\varphi'' + \varphi'^2 - \varphi'\Lambda' + \Psi'' + \Psi'^2 - \Psi'\Lambda' + \Psi'\varphi'\right) \tag{32}$$

$$8\pi p = e^{-2\Lambda}\left(\varphi'' + \varphi'^2 - \varphi'\Lambda' - \frac{\Lambda'}{r} + \frac{\varphi'}{r}\right) \tag{33}$$

We have only four equations (30) - (33) for the five unknown functions $\rho(r)$, $p(r)$, $\varphi(r)$, $\Lambda(r)$ and $\Psi(r)$.

In the following, we provide a formalism for obtaining all static cylindrically symmetric perfect solutions which depends on the specification of a single solution generating function. However, not all specifications of the generating function can generate physically acceptable solution. For obtaining physically acceptable solutions, the generating function is required satisfy some conditions.

## 5. GENERATION OF ALL STATIC CYLINDRICALLY SYMMETRIC PERFECT FLUID SOLUTIONS

From equations (31) – (33) we obtain

$$\Psi'' + \Psi'^2 + \Psi'\varphi' - \frac{\varphi'}{r} - \Psi'\Lambda' + \frac{\Lambda'}{r} = 0$$

$$\varphi'' + \varphi'^2 - \Psi'\varphi' - \frac{\psi'}{r} - \varphi'\Lambda' - \frac{\Lambda'}{r} = 0$$

These can be rearranged as

$$\Psi'' = \frac{\varphi' - \Lambda'}{r} - (\varphi' - \Lambda')\Psi' - \Psi'^2 \tag{34}$$

$$\varphi'' = \frac{\Psi' + \Lambda'}{r} + (\Psi' + \Lambda')\varphi' - \varphi'^2 \tag{35}$$

Equations (34) and (35) are Riccati type differential equations. Also $\Psi' = \frac{1}{r}$ is a particular solution of (34) for any choice of $\varphi' - \Lambda'$. The general solution of a Riccati type differential equation can be obtained if one particular solution is known. Hence the general solution of (34) can be found. Equation (35) has no such particular solution.

From equations (34) and (35) we obtain

$$\Lambda' = \varphi' - \frac{r\Psi'' + r\Psi'^2}{1 - r\Psi'} \tag{36}$$

$$\Lambda' = \frac{r\varphi'' + r\varphi'^2}{1 + r\varphi'} - \Psi' \tag{37}$$

From equations (36) and (37) we obtain

$$\frac{\varphi' - r\varphi''}{1 + r\varphi'} = \frac{r\Psi'' - \Psi' + 2\Psi'^2}{1 - r\Psi'} = f(r), \text{ say} \tag{38}$$

From (38) we obtain the pair of equations

$$\varphi'' + \frac{rf - 1}{r}\varphi' = -\frac{f}{r} \tag{39}$$

$$\Psi'' + \frac{rf - 1}{r}\Psi' = \frac{f}{r} - 2\Psi'^2 \tag{40}$$

From (36), (37) and (38) it can be seen that $\Lambda'$ is given by

$$\Lambda' = \varphi' - \Psi' - f(r) \tag{41}$$

Given f(r), (39) can be solved for $\varphi'$ as it is linear in $\varphi'$. Since $\Psi' = \frac{1}{r}$ is a particular solution of the of the Riccati type differential equation (40), it can also be solved. Knowing $\varphi'$, $\Psi'$ and f(r), $\Lambda'$ is determined by (41). $\rho(r)$ is then obtained from (30) and p(r) is obtained from any one of the equations (31) – (33). Thus, any specification of f(r) generates a solution. Conversely for any solution there is an f(r). This provides formalism for generating all static cylindrically symmetric perfect fluid solutions of Einstein's equations. Although any specification of f(r) generates a solution, it is not guaranteed that the resulting solution is physically acceptable. For physical acceptability, it is necessary that $\varphi(0) = \Psi(0) = \Lambda(0) = 0$ so that the metric coefficients are equal to 1 along the axis r = 0. We also require $\Psi'(0) = \varphi'(0) = 0$ to obtain smooth solutions along r = 0. From (4.38) we see that if $\varphi''(0)$ and $\Psi''(0)$ are finite then

$\Psi'(0)=\varphi'(0)=0$ implies f(0) = 0. In that case (4.41) shows that $\Lambda'(0)=0$. Therefore a class of physically acceptable solutions may be generated by choosing f(r) such that f(0) = 0. Inspired by this, we now provide the following algorithm for generating physically acceptable solutions.

## THE ALGORITHM

Let $f(r)=\dfrac{g'(r)}{g(r)}$

Then f(0) = 0 iff $g'(0)=0$ and g(0) $\neq$ 0. Then the solutions of (39) and (40), can be written in closed form as follows,

$$\varphi'(r)=\frac{r}{g}\left[c-\int\frac{g'(r)}{r^2}dr\right] \tag{42}$$

$$\Psi'(r)=\frac{2r^2g\int\frac{dr}{r^3g}+1}{2r^3g\int\frac{dr}{r^3g}} \tag{43}$$

Using (42) and (43) in (41) we obtain

$$\Lambda'(r)=\frac{r}{g}\left[c-\int\frac{g'(r)}{r^2}dr\right]-\frac{2r^2g\int\frac{dr}{r^3g}+1}{2r^3g\int\frac{dr}{r^3g}}-\frac{g'(r)}{g(r)} \tag{44}$$

Given g(r) such that $g'(0)=0$, $g(0)\neq 0$ the metric functions $\varphi(r)$, $\Psi(r)$ and $\Lambda(r)$ are determined by (42), (43) and (44) respectively. The energy density $\rho(r)$ and pressure p(r) are then given by (30) and (31).

Using the algorithm outlined above, we have found a new solution in Section-6.

## 6. NEW SOLUTION

Let us choose

g (r) = k = constant.

In this case we have $g'(r)=0$. Inserting these in (42), (43) and (44) we obtain

$$\varphi'(r)=ar,\quad \Psi'(r)=\frac{cr}{cr^2-1},\quad \Lambda'(r)=ar-\frac{cr}{cr^2-1}$$

$$\varphi(r)=\frac{ar^2}{2}+b_1,\quad \Psi(r)=\frac{1}{2}\log(1-cr^2)+b_2,$$

$$\Lambda(r)=\frac{1}{2}\left[ar^2-\log(1-cr^2)\right]+b_1-b_2$$

The conditions $\varphi(0)=\Psi(0)=\Lambda(0)=0$ are satisfied if $b_1=b_2=0$. Clearly $\Psi'(0)=\varphi'(0)=0$. Hence, the solution is regular on the axis of symmetry. Energy $\rho(r)$ and pressure p(r) are calculated as

$$\rho(r)=\frac{3c-a}{8\pi e^{ar^2}}$$

$$p(r)=\frac{a-c-2acr^2}{8\pi e^{ar^2}}$$

Both $\rho(r)$ and p(r) are decreasing functions of r.

$$\rho(0)=\frac{3c-a}{8\pi},\quad p(0)=\frac{a-c}{8\pi}$$

Clearly $\rho(0)>0$ and $p(0)>0$ if $c<a<3c$. Let r = R be the point where p(r) = 0

$$\Rightarrow a-c-2ac-2acR^2=0.$$

This gives

$$R=\sqrt{\frac{a-c}{2ac}}>0$$ if a > 0, c > 0 and a > c, or if a <0, c <0 and a > c.

Therefore this gives a realistic solution.

## 7. CONCLUSIONS

Choice of gauge (coordinate system) plays an important role in cylindrically symmetric solutions of Einstein's equations. We have found it convenient to use the tangential gauge. Using this gauge, we have found a new static cylindrically symmetric perfect fluid solution. Generation all such solutions depends on the specific of a single input function f(r). A class of realistic solutions is generated by choosing $f(r)=\frac{g'(r)}{g(r)}$ where g(r) = constant.

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